

Calculus III: Coursework 2
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The deadline is 5PM on Tuesday Oct 10th. Hand-in is to the green box in the basement of the Maths building.

1. Give the common name for the curve

$$x = \phi \cos \phi \quad y = \phi \sin \phi$$

with a brief explanation or sketch. [The common name should be obvious from a sketch.]

Show that arc length along this curve is

$$\int \sqrt{1 + \phi^2} d\phi$$

and then solve the integral. [Try the substitution $\phi = \sinh u$; the identity $\cosh 2u = 2 \cosh^2 u - 1$ is also useful.] [20]

Answer

The curve is a spiral. (It is like a circle, but with increasing radius as you increase ϕ .)
The differential arc length is

$$\sqrt{\left(\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2\right)} d\phi = \sqrt{1 + \phi^2} d\phi.$$

On substituting $\phi = \sinh u$ the integral becomes

$$\int \cosh^2 u du = \frac{1}{2} \left(u + \frac{1}{2} \sinh 2u \right)$$

or alternatively

$$\frac{1}{2} \left(\sinh^{-1} \phi + \phi \sqrt{1 + \phi^2} \right).$$

Marks Some reasonable attempt to sketch or explain(5), recognizing a spiral correctly (5), deriving the integral (5), solving the integral (5).

2. Derive the optimum curves which minimize $\int_0^1 F dx$ and pass through the points (0, 0), (1, 2) where F has the following forms

(a) $F = x^2 + y'^2$

(b) $F = y^2 + y'^2$

(c) $F = y'^2 + x^3 y$.

Answer

(a) F is independent of y so $\partial F/\partial y' = \text{constant}$.

This gives $y' = \text{constant}$ so $y = ax + b$ (a, b constants) and fitting the endpoints we get $y = 2x$.

(b) F is independent of x so

$$y' \frac{\partial L}{\partial y'} - L = \text{constant}.$$

This gives $y'^2 - y^2 = A^2$ where A is a constant.

$$\begin{aligned} \int \frac{dy}{\sqrt{A^2 + y^2}} &= dx \\ \Rightarrow y &= A \sinh(x + B) \end{aligned}$$

for some constant B . Passing through $(0, 0)$ implies $B = 0$. Passing through $(1, 2)$ implies $A = 2/\sinh 1$. So

$$y = \frac{2 \sinh x}{\sinh 1}.$$

(c) Here we do not have a first integral immediately. The Euler-Lagrange equation is

$$\frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) - \frac{\partial L}{\partial y} = 2 \frac{d}{dx} y' - x^3 = 2y'' - x^3 = 0.$$

Hence

$$2y' = A + x^4/4 \Rightarrow 2y = Ax + B + x^5/20$$

for some constants A and B . Passing through $(0, 0)$ implies $B = 0$. Passing through $(1, 2)$ implies $4 = A + 1/20$ i.e. $A = 79/20$, so

$$y = \frac{x^5 + 79x}{40}.$$

Marks For (a), 4 for method, 5 for accuracy; for (b) 4 and 7, for (c) 4 and 6.

3. [The brachistochrone] Consider a wire in the shape of the curve $y(x)$. If we measure y downwards, the speed of a body released at $(0, 0)$ and sliding down without friction is $\sqrt{y}$. What is the time taken for such a body to get to some point $y(x_1)$ along the wire? Derive an Euler-Lagrange equation for a shape that extremizes the time. Verify that a cycloid satisfies that equation. [You may need to look up the equation of a cycloid.] [20]

Answer

The time is

$$\int_0^{x_1} \frac{\sqrt{1 + y'^2}}{\sqrt{y}} dx$$

Minimization is of the form

$$\delta \int_{x_0}^{x_1} \mu(y) \sqrt{1 + y'^2} dx = 0$$

for which we know

$$\frac{\mu(y)}{\sqrt{1 + y'^2}} = \text{const}$$

We can write

$$y(1 + y'^2) = \text{another const}$$

For a cycloid

$$x = a(u - \sin u), \quad y = a(1 - \cos u)$$

where a is a constant, $y = 2a \sin^2 \frac{1}{2}u$ and $dx = y du$, $dy = 2a \sin \frac{1}{2}u \cos \frac{1}{2}u du$ so $1 + y'^2 = \csc^2 \frac{1}{2}u$ and hence

$$y(1 + y'^2) = 2a$$

and the solution works.

Marks Correct action (5), correct first integral (5), writing down a cycloid correctly (5), verifying the solution (5).

4. A light ray in a medium of variable composition has speed $v(y) = \cos y$ in the range $0 \leq y \leq \pi/4$. Show that (assuming light travels on the path of shortest time) the ray between the points $(0, \pi/6)$ and $(1, 0)$ in the (x, y) plane satisfies

$$2 \sinh 1 \sin y = \sinh(1 - x).$$

[Hint: for the final integration you may find a substitution of the form $\sin y = B \sinh w$, for some constant B , useful] [30]

Answer

The time taken is $\int \frac{\sqrt{1+y'^2}}{v} dx$.

As for the Fermat principle examples we get

$$\frac{1}{v\sqrt{1 + y'^2}} = C$$

where C is a constant. Inserting $v = \cos y$, and rearranging, we get

$$\int dx = \int \frac{C \cos y}{\sqrt{1 - C^2 \cos^2 y}} dy$$

The suggested substitution will give:

$\cos y dy = B \cosh w dw$ and

$1 - C^2 \cos^2 y = 1 - C^2 + B^2 C^2 \sinh^2 w$,

so the good choice is $B^2 C^2 = 1 - C^2$ making $1 - C^2 \cos^2 y = B^2 C^2 \cosh^2 w$. Then the y integration becomes

$$\int \frac{BC \cosh w \, dw}{BC \cosh w} = \int dw$$

so that $w = x + a$ for some constant a , and $\sinh(x + a) = B \sin y$.

To pass through $(1, 0)$ we need $\sinh(1 + a) = 0$ so $a = -1$.

To pass through $(0, \pi/6)$ we need $\sinh(-1) = B/2$ so $B = 2 \sinh(-1) = -2 \sinh 1$, and finally $\sinh(x - 1) = -2 \sinh 1 \sin y$ or

$$2 \sinh 1 \sin y = \sinh(1 - x).$$

Marks Correct action (5), correct first integral (5), correct substitution (5), choosing B wisely (5), using boundary conditions (8), final outcome (2).